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LETTER TO THE EDITOR

An exact solution of the Schrödinger equation for a multiterm potential

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Abstract. We show that the Schrödinger equation with the potential $Bx + Cx^2 + Dx^3 + Ex^4$ is exactly solvable on the half-line $x \geqslant 0$, provided two simple relations between B, C, D and E hold. Some remarks concerning the $B = D = 0$ case are made.

The exact solubility of the non-relativistic Schrödinger equation for various multiterm potentials is a subject which has recently attracted some attention (Quigg and Rosner 1979, Flessas and Das 1980, Johnson 1980, Flessas 1981a, b, Magyari 1981). Interest in these anharmonic oscillator-like interactions stems from the fact that, in many cases, the study of the relevant Schrödinger equation, for instance in atomic and molecular physics or models in the charmonium system, provides us with insight into the physical problem in question. Moreover, in other cases non-relativistic exact results can be used for an extrapolation into the relativistic regime, for example in the context of the theory of the S-matrix of strong interactions (de Alfaro and Regge 1965).

In this contribution we present a class of exact solutions and eigenvalues for the Schrödinger equation on the real half-axis $x \geqslant 0$

$$(\mathrm{d}^2/\mathrm{d}x^2 + \varepsilon - V(x))y(x) = 0, \tag{1}$$

ε being the eigenvalue and

$$V(x) = Bx + Cx^2 + Dx^3 + Ex^4, \qquad C > 0, \quad E > 0. \tag{2}$$

A particular feature of $V(x)$ in equation (2), a feature not possessed by other anharmonic potentials studied so far, is that it generates solutions which behave asymptotically as those for the $B = D = 0$ case:

$$V_1(x) = Cx^2 + Ex^4. \tag{3}$$

The asymptotic behaviour of the solutions for $V_1(x)$ is set out by Simon (1970). This might have been expected since in both $V(x)$ and $V_1(x)$ the highest term is x^4. Consequently, rigorous results for equation (2) may prove to be an appropriate starting point for the construction of analytic solutions to $V_1(x)$ which is of considerable importance in many investigations (Simon 1970).

To find a suitable transformation for equation (1) we invoke a theorem (Kamke 1967) regarding the behaviour of $y(x)$ for $x \to \infty$ in $(\mathrm{d}^2/\mathrm{d}x^2 + g(x))y(x) = 0$, if $g(x) \to -\infty$ as $x \to \infty$. Then, namely,

$$|(\mathrm{d}^2 y(x)/\mathrm{d}x^2)/y(x)| \to \infty \qquad \text{as } x \to \infty. \tag{4}$$

Equations (1)–(2) immediately show that the theorem applies here. Using now equation (4) we may ascertain after some algebra that a possible ansatz is

$$y(x)=f(x)\exp(ax+bx^2+cx^3), \qquad \mathrm{d}f(x)/\mathrm{d}x\to 0 \text{ as } x\to\infty, \tag{5}$$

with b and c being given by

$$9c^2=E, \qquad 12bc=D \tag{6}$$

and a as yet unknown. Introducing equation (5) into equation (1), we obtain a differential equation for $f(x)$ which can be solved by examining its structure for $x\to\infty$. The result is

$$f(x)=\exp(-\tfrac{1}{2}x)\left\{K_1+K_2\int_0^x \exp[(1-2a)t-2bt^2-2ct^3]\,\mathrm{d}t\right\}, \tag{7}$$

K_1, K_2 being constants, provided

$$2b=6c+4ab-B \tag{8}$$

$$3c=4b^2+6ac-C \tag{9}$$

$$\varepsilon=-2b-(a-\tfrac{1}{2})^2. \tag{10}$$

So we can distinguish two cases (cf equation (6)):

(i) $$c=-E^{1/2}/3. \tag{11}$$

Thus

$$b=-D/4E^{1/2}, \qquad a=\tfrac{1}{2}-(2E+BE^{1/2})/D, \tag{12}$$

$$\varepsilon=\frac{D}{2E^{1/2}}-\left(\frac{2E+BE^{1/2}}{D}\right)^2, \tag{13}$$

while between B, C, D, E the relation

$$C=\frac{D^2}{4E}+\frac{2EB+4E^{3/2}}{D} \tag{14}$$

must hold. Further, by utilising the mean-value theorem, we can assert the existence of a δ, $0\leqslant\delta\leqslant x$, such that the integral $I(x)$ in equation (7) can be written as

$$I(x)=\exp[(1-2a)\delta-2b\delta^2]\int_0^x \exp(-2ct^3)\,\mathrm{d}t. \tag{15}$$

The integral in equation (15) is essentially the confluent hypergeometric function (Gradshteyn and Ryzhik 1965). Letting $x\to\infty$ and taking into account equations (7), (11), (15) as well as the asymptotic formula for the confluent hypergeometric function, we deduce that

$$I(x)\simeq\exp[(1-2a)x_1-2bx_1^2+(2E^{1/2}/3)x^3]x^{-2}, \qquad x\to\infty, \tag{16}$$

$0\leqslant x_1\leqslant\infty$ (x_1 stands for δ when in equation (15) we set $c=-E^{1/2}/3$). Equation (16) shows that in equation (7) we have to choose $K_2=0$ to ensure $y(x)\to 0$ as $x\to\infty$ in equation (5). Since we are interested in symmetric potentials, which implies that for $x\leqslant 0$ the corresponding interaction is derived from equation (2) by replacing B and D with $-B$ and $-D$, respectively, we impose on $y(x)$ the conditions $y(0)=1$ and

$(\mathrm{d}y(x)/\mathrm{d}x)_0 = 0$ which, for our case, are suitable for *even*-parity solutions. Therefore we finally obtain the physically acceptable solution to equation (1):

$$y(x) = \exp\left(-\frac{D}{4E^{1/2}}x^2 - \frac{E^{1/2}}{3}x^3\right), \qquad x \geqslant 0 \tag{17}$$

provided, as follows from equations (12)–(14), equation (5), equation (7) and $(\mathrm{d}y(x)/\mathrm{d}x)_0 = 0$,

$$B = -2E^{1/2}, \qquad C = D^2/4E, \qquad \varepsilon = D/2E^{1/2}. \tag{18, 19}$$

The above stated similarity of $y(x)$ to the solutions for $V_1(x)$ in equation (3) becomes evident from equation (17).

(ii) $$c = E^{1/2}/3. \tag{20}$$

First, we get the solution for $x \leqslant 0$ which is equivalent to equation (17) and $V(x) = -Bx + Cx^2 - Dx^3 + Ex^4$, where B, C and ε of course again satisfy equations (18)–(19):

$$y(x) = \exp\left(-\frac{D}{4E^{1/2}}x^2 + \frac{E^{1/2}}{3}x^3\right), \qquad x \leqslant 0. \tag{21}$$

Second, returning to the treatment of $V(x)$ in equation (2), one can write down the relations corresponding to equations (12)–(14) and in a similar way arrive at the following solution to equation (1) which vanishes at infinity and satisfies $y(0) = 1$:

$$y(x) = -\frac{1}{I(\infty, \alpha)} \exp\left[-\left(\frac{2E - BE^{1/2}}{D}\right)x + \frac{D}{4E^{1/2}}x^2 + \frac{E^{1/2}}{3}x^3\right]$$
$$\times\left\{-I(\infty, \alpha) + \int_0^x \exp\left[\left(\frac{4E - 2BE^{1/2}}{D}\right)t - \frac{D}{2E^{1/2}}t^2 - \frac{2E^{1/2}}{3}t^3\right]\mathrm{d}t\right\},$$
$$x \geqslant 0, \tag{22}$$

$$I(\infty, \alpha) = \int_0^\infty \exp\left(-\frac{2\alpha}{D}t - \frac{D}{2E^{1/2}}t^2 - \frac{2E^{1/2}}{3}t^3\right)\mathrm{d}t$$
$$\alpha = -2E + BE^{1/2} \tag{23}$$

provided

$$C = D^2/4E + 2\alpha E^{1/2}/D, \qquad \varepsilon = -D/2E^{1/2} - \alpha^2/D^2. \tag{24}$$

The boundary condition $(\mathrm{d}y(x)/\mathrm{d}x)_0 = 0$ leads to the constraint

$$D/\alpha = I(\infty, \alpha), \tag{25}$$

which shows that D and α must have the same sign. If $D > 0$ then, as

$$\frac{D}{\alpha} > \frac{D}{2\alpha} = \int_0^\infty \exp\left(-\frac{2\alpha}{D}t\right)\mathrm{d}t > I(\infty, \alpha),$$

equation (25) cannot be satisfied. If on the other hand $D < 0$, then also $\alpha < 0$ and as the left-hand side of equation (25) increases linearly with $-D$, while the right-hand side increases exponentially, equation (25) again cannot be satisfied. The same result is obtained if one tries to calculate α from equation (25). Thus we are left with the rigorous system of equations (17)–(19) for $x \geqslant 0$ and equation (21) for $x \leqslant 0$. This fact taken in conjunction with the similarity which exists, in the sense mentioned above, between equations (2) and (3) allows us to formulate the hypothesis that possibly some of the physics which pertains to the potential (3) might be reproduced by looking at the interaction (2) and the analogous one for $x \leqslant 0$. Finally, as has now been shown for the solutions of Flessas and Das (1980) by Khare (1981), the solutions presented here possess an interesting feature, namely in the limit $D \to 0$, $E \to 0$ with $D < 0$ and (cf equation (18)) $D^2/4E =$ finite $= C$, they are inaccessible to conventional perturbation theory around the harmonic term since the correct energy is (from equation (19)) $\varepsilon = -C^{1/2}$ and *not* $C^{1/2}$.

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